Professional paper
Accepted 30. 11. 2011.

MARTINA TRIPLAT HORVAT
MILJENKO LAPAINE
DRAŽEN TUTIĆ

Application of Bošković's Geometric Adjustment Method on Five Meridian Degrees

On the Occasion of 300th Anniversary of the Birth of Ruđer Josip Bošković

Application of Bošković's Geometric Adjustment Method on Five Meridian Degrees

ABSTRACT

In this paper, the first method of adjustment, proposed by Josip Ruđer Bošković, is described in detail, on the example on five meridian degrees. Bošković sets three conditions on the data of the lengths of the meridian degrees to calculate corrections that would fix all degrees in order to get a better estimate of true values. The conditions that have to be satisfied are explained by geometric method which Bošković described in all his studies. For the purpose of this paper, in the process of computing these five meridian degrees, data from Bošković original book have been used.

Geometric solution, described by Bošković himself, is not easy to understand at first, as this is noted by other authors who have studied Bošković's method as well. Hence, geometric description of the Bošković's method is shown in analytical form as well.

Key words: Josip Ruđer Bošković, geometric adjustment method

MSC 2010: 01A50, 86A30, 62-03, 51-03, 62A01, 62P99, 62J05, 41A10

Primjena Boškovićeve geometrijske metode izjednačenja na pet meridijanskih stupnjeva

SAŽETAK

U ovom radu detaljno je prikazana prva metoda izjednačenja, koju je osmislio Josip Ruđer Bošković, na primjeru pet stupnjeva meridijana. Bošković je izračunao popravke kojima bi popravio duljine meridijanskih stupnjeva i na taj način dobio što bolje procjene njihovih pravih vrijednosti. Postavljajući tri uvjeta tom prilikom formirao je svoju metodu izjednačenja koju je primijenio na podatke o duljinama meridijanskih stupnjeva. Uvjeti koji moraju biti zadovoljeni objašnjeni su geometrijskom metodom kakvu Bošković opisuje u svim svojim djelima. U postupku računanja, koja su provedena u ovom radu na pet meridijanskih stupnjeva, korišteni su podaci iz Boškovićevih originalnih djela.

Geometrijsko rješenje kako ga je Bošković opisao nije odmah lako razumljivo, što su uočili i drugi autori koji su proučavali Boškovićevu metodu. Stoga će geometrijski opisana Boškovićeva metoda biti također prikazana i u analitičkom obliku.

Ključne riječi: Josip Ruđer Bošković, geometrijska metoda izjednačenja

1 Introduction

Ruđer Josip Bošković (Dubrovnik, 18th May 1711 - Milan, 13th February 1787) began to publish theses on Earth's shape and size as a young scientist. These issues were a major scientific problem of the 18th century. In 1739, when he was only 28, he published two dissertations: *De veterum argumentis pro telluris sphaericitate* (On the arguments of the ancients for the sphericity of the Earth) and *Dissertatio de telluris figura* (A dissertation on the shape of the Earth).

During the 18th century scientists were having a great discussion about the question whether the Earth was oblate or oblong (prolate) at the poles. In the late 17th century, Newton proved that the Earth should be flattened at the poles because of its rotation. Domenico Cassini assumed the opposite, that Earth had the shape of an egg so, at the end of

17th and the beginning of 18th century he conducted comprehensive geodetic observations to prove his assumption.

During this period there were two basic methods for determining the Earth's figure: pendulum experiments and the determination of the meridian arc length. The idea of the second method was to determine the length of the meridian arc that corresponded to one degree of latitude. French Academy carried out the measurements during 1730s to test theoretical interpretations of the Earth's figure.

2 Bošković's Thoughts on the Shape of the Earth

Bošković thought that the irregularity of the shape of the Earth could be examined in the way to "exactly determine two meridian degrees in different longitudes, but in the same latitude" [1]. Bošković wanted to measure a meridian degree at some latitude, on which another meridian degree is accurately determined, but at a different longitude. Pope Benedict XIV gave, Cardinal Valenti, the State Secretary of the Holy See, permission for Bošković to perform "astronomical and geographical journey" along the meridian from Rome to Rimini in the Papal State because Rome and Rimini are approximately at the same meridian. The area comprised of two meridian degrees (Fig. 1). The length of the middle meridian degree, between Rome and Rimini, could be compared with the length of the measured meridian degree in the south France. This length of the meridian degree was measured ten years earlier in Perpignan, by Jacques Cassini and Nicolas Louis de Lacaille (De la Caille). In Fig. 1, the labels A, B, C, D, E, F, G, H, I and L show points of the chain of triangles between Rome and Rimini with the names of the hills on which points are located.

Bošković chose Christopher Maire for companion. In 1755 Bošković and Maire published the first results of those measurements and the analysis of measured data in the book *De Litteraria Expeditione per Pontificiam ditionem ad dimentiendas duas Meridiani gradus et corrigendam mappam geographicam* (A scientific journey through the Papal State with the purpose of measuring two degrees of meridian and correcting a geographical map) in more than 500 pages.

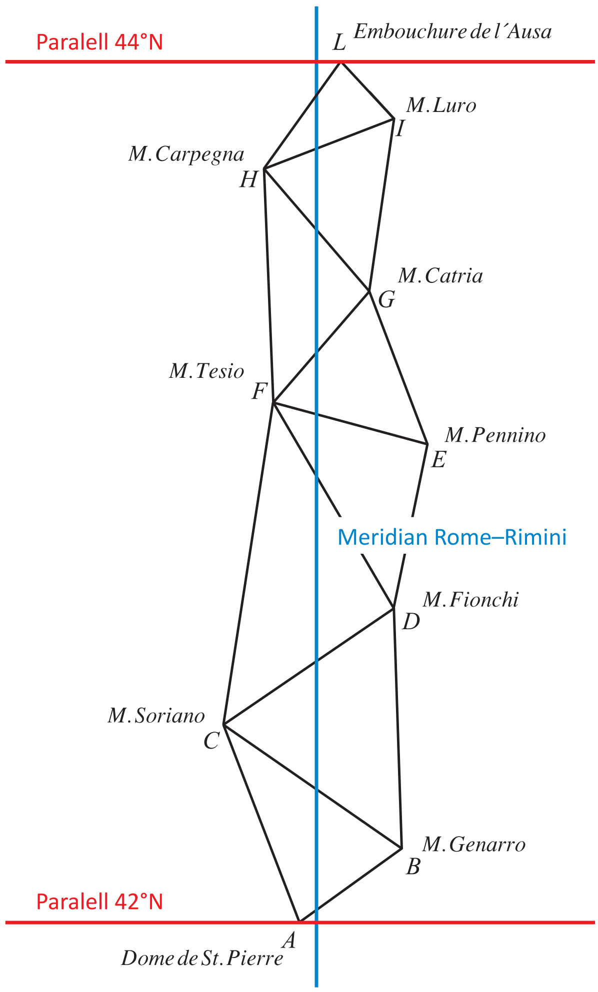

Figure 1: *Arc of the meridian Rome-Rimini and the system of the triangle chain between them (as in Fig. 2 in the first annex from the book* Voyage astronomique ...*).*

Bošković published his results of geodetic measurements three more times:

- in 1757, in the abstract form for the journal of the Academy of Bologne
- in 1760, in the supplement of the Benedict Stay's poem and
- in 1770, in French translation of his major geodetic work *Voyage astronomique...* in which he gave an example of adjustment.

In 1760 Bošković started processing the results of measurements of meridian degrees that were conducted after 1736.

In order to accurately determine the figure of the Earth, Bošković, in his first attempt to determine ellipticity, compared five arc lengths of one meridian degree, which he considered to be sufficiently accurate and in his second attempt he compared nine meridian degrees. The measurements of the five meridian degrees were carried out in Quito, in the Cape of Good Hope, in Paris, in Lapland and his own, carried out in Rome, Italy [2].

Whereas astronomical and geodetic measurements are liable with errors caused by various sources, Bošković was aware that the causes of errors could not be fully eliminated during the construction of instruments and measurements. When comparing mentioned degrees of meridian, Bošković could not determine such an ellipsoid consistent with all the measurements. He decided to determine corrections that would fix all degrees and get a better estimate of true values. He formed his own adjustment method of the results of measurements proposing three conditions for determining the corrections.

3 Bošković's Adjustment Method

In his works Bošković presented only geometric approach to obtain corrections. In the summary of his main book [1] Bošković stated that he used algebraic approach only to derive short formulas based on the geometric solution, which immediately gave the solution.

Bošković has set the problem in the following way. It is necessary to calculate the mean ellipticity (fr. *ellipticit*, lat. *ellipticitate*) of all meridian degrees, which are mutually compared. Taking into account the relation that has to have differences of (compared) meridian degrees as well as the laws of probabilities regarding to corrections, it is necessary to adjust the degrees to be reduced to this relation. To obtain such mean, which is not simply arithmetic mean, but tied by certain law of fortuitous combinations and the calculus of probabilities, and considering a certain number of meridian degrees, the corrections have to be found and applied to each measured meridian degree, taking into account the following three conditions [1], [3], [5]:

1. The differences of the meridian degrees are proportional to the differences of the versed sines[1] of double latitudes
2. The sum of the positive corrections is equal to the sum of the negative ones (by their absolute values) and
3. The absolute sum of all the corrections, positive as well as negative, is the least possible one, in case in which first two conditions are fulfilled.

Bošković formed the first condition out of the requirements of the law of balance, or the assumption that the Earth had the shape of an ellipsoid. Starting from the laws of gravity, Isaac Newton, in the second half of the 17th century, claimed that as the result of the Earth's rotation around its axis and the mutual attraction between the planets's mass, Earth should be flattened at the poles. Newton (1726) [4] in his *Principia* in Volume 3, Proposition 20 says: "Whence arises this Theorem, that the increase of weight in passing from the equator to the poles is nearly as the versed sine of double the latitude; or, which comes to the same thing, as the square of the right sin of the latitude; and the arcs of the degrees of latitude in the meridian increase nearly in the same proportion.".

The second condition emerged from the fact that the deflections of the pendulum or observer's errors that increase or decrease the meridian degrees have the same degree of probability, or the errors with positive and negative signs are equally probable. To fulfil the second condition, the sum of all the values of corrections should be equalized to zero and Bošković said that this was the only condition in which the sum of the positive can be equalized with the sum of the negative ones.

The third condition is necessary to approximate the measured values as much as possible because the measurement errors are probably very small. Also, this third condition, Bošković proposed because the solution was not completely defined with the first two conditions [3], [5].

In the process of computation, performed for the purpose of this paper, measured values shown in Table 1 are used. The arc lengths of meridian and their corresponding latitudes (2nd and 4th column of Table 1) have been taken out of Bošković's original books [1], [3], [5]. Other values from the Table 1 are calculated from these values. Instead of versed sines of double latitudes, a half of their values are taken (according to [2]).

[1] versed sines - reversed sin, a trigonometric function of an angle or arc which are not in use today, and is defined as $\mathrm{versin}\varphi = 1 - \cos\varphi$.

Place of the measured meridian degree	**Latitude of the measured degree**	1/2versin **multiplied by 10 000**	**Arc length [toise]**	**Differences from the first degree measured in Quito [toise]**
Quito	0° 0'	0 (*AA*)	56 751 (*Aa*)	0 (*aa*)
Cape of Good Hope	33° 18'	3014 (*AB*)	57 037 (*Bb*)	286 (*Ob*)
Rome	42° 59'	4648 (*AC*)	56 979 (*Cc*)	228 (*Pc*)
Paris	49° 23'	5762 (*AD*)	57 074 (*Dd*)	323 (*Qd*)
Lapland	66° 19'	8387 (*AE*)	57 422 (*Ee*)	671 (*Re*)

Table 1: *Five measured meridian degrees and their latitudes (names of places and the values in the second and fourth columns are taken from* [3]*).*

Bošković's procedure for determining the corrections can be described as following. Versed sines of double latitudes of five meridian degrees are put in a rectangular coordinate system on the abscissa (Fig. 2). The points B, C, D and E are drawn as if they were measured at the equator from the point A. On the vertical axis aA, bB, cC, dD and eE segments are put. They represent the five lengths of the corresponding measured arcs (in toises[2] per degree) that need to be fixed. These segments are perpendicular to AF. The size of the AF can be considered as one unit in length, while the A, B, C, D and E are represent the five values of the $sin^2\varphi$ marked on the unit interval. Equator passes through point A, and the North Pole through point F.

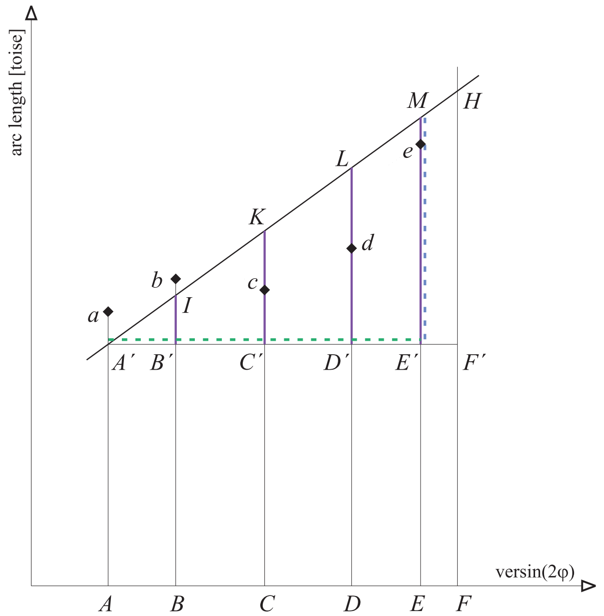

Figure 2: *Proportionality for the degrees.*

Bošković said that any straight line which intersects those segments could determine a degree which would satisfy the first condition. If we draw a straight line $A'F'$ through the point A', which is parallel to AF, the determined meridian degrees will intersect that straight line at points B', C', D' and E'. Values $E'M$, $D'L$, $C'K$, $B'I$ and $A'A'$ (zero), given in such way, represent the differences of the degrees according to a degree at the equator and they are proportional to the values $A'E'$, $A'D'$, $A'C'$, $A'B'$ and $A'A'$ (zero) or a versed sines AE, AD, AC, AB and AA (zero) [3]. The proportionality for the measured degree at the point e with dashed (blue and green) lines is shown in Figure 2.

Bošković's first condition can be expressed analytically in such way that the corrections can be expressed with the equation of the first degree which includes two (unknown) sizes, values k and l, as well as the degrees of meridian and corresponding versed sines of double latitudes:

$$L_i + v_i = k\text{versin}2\varphi_i + l, \quad (i = 1, 2, \cdots, n), \tag{1}$$

where is

l - the length of a meridian degree at the equator

k - the excess of a degree at the pole over one at the equator,

L_i - the length of an arc at location i

φ_i - the latitude of the midpoint of the arc at location i

v_i - the corrections of a meridian degree.

In order to determine the first point of the required straight line the second condition, that the sum of the positive corrections is equal to the sum of the negative ones (by their absolute values), will be applied.

The ordinate segments eM, dL, cK, bI and aA' are corrections with positive (blue segments in Fig. 3) or negative sign (red segments in Fig. 3). Their sign depends on whether the points e, d, c, b and a are located on one or the other side of straight line $A'H$ in relation to AF. The second condition will be satisfied when the line passes through the common centre of gravity G of those points. According to the centre of the gravity, the sum of the distances of all points on one side, in any direction, is equal to the

[2]Toise - the old measure unit which are not in use today, 1 toise = 1.949 m.

sum of the distances of all points on the opposite side of the straight line [3].

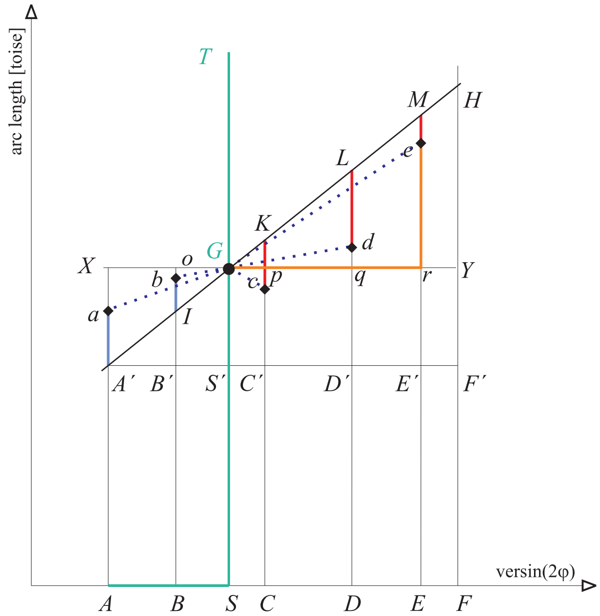

Figure 3: *Simplified Fig. 7 from Bošković's first annex with auxiliary sizes needed for calculation.*

The coordinates of the barycentre G are determined by the values AS and SG (values shown in Fig. 3 with green lines). Latitude which corresponds to the position of the point G is defined by the distance AS. The value AS was calculated by dividing the sum of values in the third column of Table 1 with the number of the measurements.

$$AS = \frac{AA+AB+AC+AD+AE}{5} = 4362.2 \tag{2}$$

The value SG defines position of point G completely. This value is applied perpendicularly to AF, and is calculated as the arithmetic mean of the values that are in the fourth column of Table 1.

$$SG = \frac{Aa+Bb+Cc+Dd+Ee}{5} = 57052.6 \tag{3}$$

AS and SG define the first meridian degree which defines the straight line. Through point G the infinite number of straight lines can be drawn that will satisfy the first two conditions.

Analytically, second condition can be written as follows

$$\sum_{j=1}^{m} v_j^+ = -\sum_{k=1}^{u} v_k^-, \tag{4}$$

where
$\sum_{j=1}^{m} v_j^+$ – the sum of m positive corrections,
$\sum_{k=1}^{u} v_k^-$ – the sum of u negative corrections.

The second condition derived from the expression (4) can also be written in the following form

$$\sum_{i=1}^{n} v_i = 0, \tag{5}$$

where
$n = m + u$ – is the number of measurements.

In order to fully determine the straight line it is necessary to determine one more point. In this case we can use the third condition.
To satisfy the third condition it is necessary to determine in which order the moving straight line, which rotates around the point G, passes each point. Visualize a line with a starting position SGT (as it is shown with light green line in the Fig. 3) that rotates clockwise around the point G. In the beginning it will close a small angle, and corrections will be big. By rotating the straight line, the absolute values of corrections will decrease until the line reaches any point a, b, c, d or e. When the line falls into any of these points, the correction in that point will be cancelled. As soon as the line passes that point, correction which corresponds to that particular point will change sign and it will begin to grow. At the same time, corrections of the other points to which line has not yet come will continue to decrease. Accordingly, the absolute sum of all corrections will decrease until the sum of the corrections that increase is not greater than the sum of the corrections that decrease. This conclusion comes from the fact that the sums of the corrections with the positive and negative signs are the same if we consider their absolute value and each of the sums contains half of the total sum.
The order in which the moving straight line passes through any point can be determined numerically. Bošković, however specifies that calculating will not be necessary. Structure itself, provided it is true, will be sufficient to determine the order in which the line passes through each point.
The line XY is drawn parallel to AF thought the point G (Fig. 3). Perpendiculars to the AF which pass through points A, B, C, D and E will intersect the line XY in o, p, q and r, respectively. If we visualize that the barycenter G is the center of a rectangular coordinate system, the angles SGY, YGT, TGX and XGS form the quadrants of this coordinate system. Firstly, it should be determined in which quadrant every point is, since each of them are left or right in relation to SGT, depending on whether their versed sines are smaller or larger than the AS. Also, each point must be below or above XGY depending on whether its degree is smaller or larger than the SG [3].

	Quito (a)	Cape of Good Hope (b)	Rome (c)	Paris (d)	Lapland (e)
A	4362.2 (AS)	1348.2 (BS)	-286.8 (CS)	-1400.8 (DS)	-4025.8 (ES)
B	301.6 (Xa)	15.6 (ob)	73.6 (pc)	-21.4 (qd)	-369.4 (re)
C=A/B	14.5	86.4	-3.9	65.4	10.9

Table 2: *Auxiliary computation for solving the third condition.*

Also the tangents of the angles which make GS or GT to the line that passes through each point (shown with dashed blue line in Fig. 3) should be found. If the line passes through the point e then the re is in relation to Gr, as the radius according to the tangent of the angle reG, or eGT. This relation can be written as follows (shown with orange lines in Fig. 3):

$$\tan reG = \frac{Gr}{re}. \tag{6}$$

Since Gr (equals to ES) is the residual of versed sines of e and versed sines of AS (values in Table 2, row A) and re is the residual of degrees Ee and SG (values in Table 2, row B), these values should be divided and used to sort a series of numbers (values in Table 2, row C). The series begins with a growing range of positive numbers (points contained in the first and the third quadrant of the coordinate system), and continues with a descending range of negative numbers (points that are contained in the second and the fourth quadrant of the coordinate system).

To determine all required values from Table 2 it is necessary to calculate the distances aX, bo, cp, dq, and er (Table 2, row B) to the straight line XY. These distances are calculated as residuals of NG and values aa, Ob, Pc, Qd and Re (values in the fifth column, Table 1). The value NG equals to the arithmetic mean of the numbers of the fifth column in Table 1 and amounts to 301.6.

Distances AS, BS, CS, DS and ES of points a, b, c, d and e in relation to the straight line SGT are equal to the difference of the size AS from the sizes AA, AB, AC, AD and AE (the third column in Table 1). The calculated distances are shown in Table 2, row A.

Tangents of the angles (Table 2, row C) are calculated as the ratio of the size of the line A and B in Table 2. Arranging a series with the resulting values from the row C, the rotating line intersects the points in the order e, a, d, b, c.

To find the position of the straight line that will satisfy the required minimum (the sum of all corrections are at least possible) the absolute values of the sizes from row A in Table 2 are summed in the mentioned order until the sum of the values do not pass half of their total sum.

The first absolute value for the point e equals 4025.8 and it is less than the half of the total absolute sum which is 5710. Then the e value is added to the following value from the series (a) 4362.2. Their sum equals 8388 which exceeds half of the total absolute sum. When the rotating straight line passes the point a, the size exceeds half of their total absolute sum. In this way we get the minimum that is required.

Analytically this condition can be written as follows:

$$\sum_{i=1}^{n} |v_i| = minimum. \tag{7}$$

By finding the straight line that satisfies the conditions defined by two meridian degrees all other degrees and their corrections can be found.

To determine the corrections bi, ck, dl and em, the value fV (the residual between the length of the degree at the pole and at the equator) needs to be calculated, using the ratio:

$$aN : NG = af : fV \Rightarrow fV = \frac{NG \cdot af}{aN} = 691.4. \tag{8}$$

Respecting the requirement of proportionality with the residuals of the sizes Oi, Pk, Ql and Rm of the Ob, Pc, Qd and Re, the corrections of certain meridian degrees can be calculated. The quantities Oi, Pk, Ql, and Rm (Table 3, column 2) are the distances from the straight line aV to the line af. The procedure for their calculation is shown on the example of the values Oi, which is obtained from the ratio:

$$af : aO = fV : Oi \Rightarrow Oi = \frac{aO \cdot fV}{af} = 208.4. \tag{9}$$

And the values Ob, Pc, Qd and Re (Table 3, column 3 and Table 1, row 5) are the differences of certain meridian degrees from the first meridian (Aa) because it stays without correction.

	Quito (a)	Cape of Good Hope (b)	Rome (c)	Paris (d)	Lapland (e)
Distances of straight lines aV i af [toise]	0	208.4	321.3	398.4	579.9
Residual from the first degree [toise]	0	286	228	323	671
Corrections [toise]	0	-77.6	+93.3	+75.4	-91.1

Table 3: *Calculated values of corrections.*

Sum of the corrections (Table 3, column 4) with a positive and with a negative sign are equal in absolute value and equals 168.7 toises which satisfies the second condition. Five meridian arc lengths and straight line that satisfies conditions are shown in Figure 4. Equation of a straight line that satisfies all conditions can be written as follows:

$$L = 0.07\text{versin}2\varphi + 56751. \qquad (10)$$

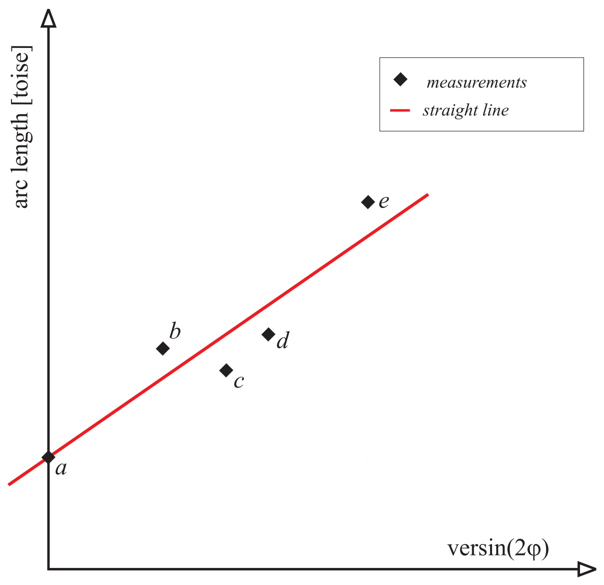

Figure 4: *Five meridian arc lengths and Bošković's straight line that satisfies conditions.*

4 Conclusions

The straight line which passes through points a and G gives the smallest sum of all corrections (satisfying the third condition), and the sum of positive and negative corrections is equal in its absolute value (satisfying the second condition). Point a remains without correction since the straight line, that gives the minimum, passes exactly through that point.

Bošković defined and applied the principle in which the measured values can be approximated with linear function. With defined conditions, unknowns and adjusted values can be determined. With equation of a straight line, Bošković linked the determined length of meridian degrees and versed sines of their corresponding latitudes, and as unknown sizes he determined values of parameters of the straight line (k and l).

In this paper we presented a method of adjustment proposed by Bošković and developed by Laplace. Today this method is mostly known as L_1-norm method. It uses only one condition (the third one) out of the three proposed by Bošković.
Difficulties in implementation of L_1-norm occurred when it was needed to adjust a greater number of unknowns. With the development of computer technology and algorithms based on linear programming this is no longer insuperable problem.

Today the most used method of adjustment is the method of least squares, also known as L_2-norm method. Bošković's adjustment method certainly cannot replace the now well established method of least squares, but this does not exclude the possibility that these two methods can be used parallel and complement each other. The most significant contribution of Bošković's method, and also its most important application is the detection of gross errors in measurements which gives excellent results.

References

[1] Boscovich, R. J. (1757): *De litteraria expeditione per pontificiam ditionem*. De Bononiensi Scientiarum et Artium Instituto atque Academia Commentarii, Tomus Quartus, 353-394. Typis Lalii a Vulpe Instituti Scientiarum Typographi, Bononiae.

[2] Maire, Ch., Boscovich, R. J. (1755): *De litteraria expeditione per pontificiam ditionem ad dimetiendos duos meridiani gradus et corrigendam mappam geographicam*. Typographio Palladis, Roma.

[3] Maire, Ch., Boscovich, R. J. (1770): *Voyage astronomique et ggraphique, dans l'tat de l'Eglise, entrepris par l'ordre et sous les auspicer du pape Benoit XIV, pour mesurer deux dgrs du meridian, & corriger la Carte de l'Etat ecclsiastique*. N. M. Tilliar, Paris.

[4] NEWTON, I. (1726): *Philosophiae Naturalis Principia Mathematica*. Translated, 1846; as Newtons Principia the Mathematical Principles of Natural Philosophy, by Sir Isaac Newton; Translated Into English by Andrew Motte, ed. Daniel Adee, New York.

[5] STAY, B. (1760): *Philosophiae recentioris a Benedicto Stay versibus tradita libri X Cum adnotationibus, et supplementis P. Rogerii Josephi Boscovich S. J.*, Tomus II. Typis, et sumptibus Nicolai et Marci Palearini, Roma.

Martina Triplat Horvat
e-mail: mthorvat@geof.hr
Miljenko Lapaine
e-mail: mlapaine@geof.hr
Dražen Tutić
e-mail: dtutic@geof.hr
Faculty of Geodesy University of Zagreb
Kačićeva 26, 10000 Zagreb, Croatia